# RIEMANNIAN GEOMETRY & GEOMETRIC ANALYSIS

**[1]DR. RAJENDRA & [2]DR. VIKAS SOLANKI**

**[1]ASSISTANT PROFESSOR, DEPARTMENT OF MATHEMATICS, DR. BHIMRAO AMBEDKAR GOVT. COLLEGE , SRIGANGANAGAR , (RAJASTHAN)**

**[2]ASSISTANT PROFESSOR, DEPARTMENT OF MATHEMATICS, DR. BHIMRAO AMBEDKAR GOVT. COLLEGE , SRIGANGANAGAR , (RAJASTHAN)**

## ABSTRACT

Riemannian geometry is the branch of differential geometry that studies Riemannian manifolds. An example of a Riemannian manifold is a surface, on which distances are measured by the length of curves on the surface. Riemannian geometry is the study of surfaces and their higher-dimensional analogs (called manifolds), in which distances are calculated along curves belonging to the manifold. Formally, Riemannian geometry is the study of smooth manifolds with a Riemannian metric (an inner product on the tangent space at each point that varies smoothly from point to point). This gives, in particular, local notions of angle, length of curves, surface area and volume. From those, some other global quantities can be derived by integrating local contributions.

Riemannian geometry originated with the vision of Bernhard Riemann expressed in his inaugural lecture "Über die Hypothesen, welche der Geometrie zu Grunde liegen" ("On the Hypotheses on which Geometry is Based").[1] It is a very broad and abstract generalization of the differential geometry of surfaces in R3. Development of Riemannian geometry resulted in synthesis of diverse results concerning the geometry of surfaces and the behavior of geodesics on them, with techniques that can be applied to the study of differentiable manifolds of higher dimensions. It enabled the formulation of Einstein's general theory of relativity, made profound impact on group theory and representation theory, as well as analysis, and spurred the development of algebraic and differential topology.

KEYWORDS- Riemannian geometry, analysis, algebraic and differential topology, integral

## INTRODUCTION

Geometric analysis is a major branch of mathematics that uses tools from modern analysis—especially partial differential equations (PDEs)—to study the shape, geometry, and topology of spaces like Riemannian manifolds. [1, 2]

Core Concepts & Methods

- Interplay of Fields: It connects geometry (shapes, curves, surfaces, and curvature) with analytic techniques (calculus of variations and differential equations). [1, 2]
- Partial Differential Equations (PDEs): Different geometric problems rely on different types of PDEs:
  - Elliptic PDEs appear in conformal geometry and minimal surface problems.
  - Parabolic PDEs (like the Ricci flow) drive evolution equations on manifolds.
  - Hyperbolic PDEs arise in general relativity and wave equations. [1, 2, 3, 4]
- Geometric Flows: Methods like the Ricci flow or mean curvature flow deform shapes over time to reveal their underlying topological structure. [1, 2, 3]

Key Historical Milestones

- Early Foundations: Pioneer work on minimal surfaces and curvature was done by figures like Jesse Douglas, Tibor Radó, John Nash, Louis Nirenberg, and Aleksandrov.
- The Poincaré Conjecture: Richard Hamilton introduced the Ricci flow program, which was later used by Grigori Perelman to solve the famous Poincaré conjecture.
- Physics Connections: The discipline heavily overlaps with mathematical physics, providing tools for general relativity (such as the positive mass theorem), string theory (harmonic mappings), and quantum field theory. [1, 2, 3, 4]

Geometric analysis is a mathematical discipline where tools from differential equations, especially elliptic partial differential equations (PDEs), are used to establish new results in differential geometry and differential topology. The use of linear elliptic PDEs dates at least as far back as Hodge theory. More recently, it refers largely to the use of nonlinear partial differential equations to study geometric and topological properties of spaces, such as submanifolds of Euclidean space, Riemannian manifolds, and symplectic manifolds. This approach dates back to the work by Tibor Radó and Jesse Douglas on minimal surfaces, John Forbes Nash Jr. on isometric embeddings of Riemannian manifolds into Euclidean space, work by Louis Nirenberg on the Minkowski problem and the Weyl problem, and work by Aleksandr Danilovich Aleksandrov and Aleksei Pogorelov on convex hypersurfaces. In the 1980s fundamental contributions by Karen Uhlenbeck,[1] Clifford Taubes, Shing-Tung Yau, Richard Schoen, and Richard Hamilton launched a particularly exciting and productive era of geometric analysis that continues to this day. A celebrated achievement was the solution to the Poincaré conjecture by Grigori Perelman, completing a program initiated and largely carried out by Richard Hamilton.[4,5,6]

Scope

The scope of geometric analysis includes both the use of geometrical methods in the study of partial differential equations (when it is also known as "geometric PDE"), and the application of the theory of partial differential equations to geometry. It incorporates problems involving curves and surfaces, or domains with curved boundaries, but also the study of Riemannian manifolds in arbitrary dimension. The calculus of variations is sometimes regarded as part of geometric analysis, because differential equations arising from variational principles have a strong geometric content. Geometric analysis also includes global analysis, which concerns the study of differential equations on manifolds, and the relationship between differential equations and topology.

The following is a partial list of major topics within geometric analysis:

- Gauge theory
- Harmonic maps
- Kähler–Einstein metrics
- Mean curvature flow
- Minimal submanifolds
- Positive energy theorems
- Pseudoholomorphic curves
- Ricci flow

- Yamabe problem
- Yang–Mills equations

## DISCUSSION

A modern theorem is that every smooth manifold admits a Riemannian metric (in fact, many Riemannian metrics). The properties of such metrics are useful to constrain the topology of the original manifold.

In Riemannian geometry, as in Euclidean geometry, the quadratic form is positive definite. Relaxing this condition, and allowing that some non-zero vectors can be null under the quadratic form allows the structure of pseudo-Riemannian manifolds, which (in four dimensions) are the main objects of the theory of general relativity. On the other hand, replacing the quadratic form by a more general non-quadratic function leads to Finsler geometry.

There exists a close analogy of differential geometry with the mathematical structure of defects in regular crystals. Dislocations and disclinations produce torsions and curvature.[2][3]

The following articles provide some useful introductory material:[7,8,9]

- Metric tensor
- Riemannian manifold
- Levi-Civita connection
- Curvature
- Riemann curvature tensor
- List of differential geometry topics
- Glossary of Riemannian and metric geometry

Classical theorems

What follows is an incomplete list of the most classical theorems in Riemannian geometry. The choice is made depending on its importance and elegance of formulation. Most of the results can be found in the classic monograph by Jeff Cheeger and D. Ebin (see below).

The formulations given are far from being very exact or the most general. This list is oriented to those who already know the basic definitions and want to know what these definitions are about.

General theorems

1. Gauss–Bonnet theorem The integral of the Gauss curvature on a compact 2-dimensional Riemannian manifold is equal to $2\pi\chi(M)$ where $\chi(M)$ denotes the Euler characteristic of M. This theorem has a generalization to any compact even-dimensional Riemannian manifold, see generalized Gauss-Bonnet theorem.
2. Nash embedding theorems. They state that every Riemannian manifold can be isometrically embedded in a Euclidean space $R^n$.

Geometry in large

In all of the following theorems we assume some local behavior of the space (usually formulated using curvature assumption) to derive some information about the global structure of the space, including either some information on the topological type of the manifold or on the behavior of points at "sufficiently large" distances.

Pinched sectional curvature

1. Sphere theorem. If M is a simply connected compact n-dimensional Riemannian manifold with sectional curvature strictly pinched between 1/4 and 1 then M is diffeomorphic to a sphere.
2. Cheeger's finiteness theorem. Given constants C, D and V, there are only finitely many (up to diffeomorphism) compact n-dimensional Riemannian manifolds with sectional curvature $|K| \leq C$, diameter $\leq D$ and volume $\geq V$.
3. Gromov's almost flat manifolds. There is an $\varepsilon_n > 0$ such that if an n-dimensional Riemannian manifold has a metric with sectional curvature $|K| \leq \varepsilon_n$ and diameter $\leq 1$ then its finite cover is diffeomorphic to a nil manifold.[10,11,12]

Sectional curvature bounded below

1. Cheeger–Gromoll's soul theorem. If M is a non-compact complete non-negatively curved n-dimensional Riemannian manifold, then M contains a compact, totally geodesic submanifold S such that M is diffeomorphic to the normal bundle of S (S is called the soul of M.) In particular, if M has strictly positive curvature everywhere, then it is diffeomorphic to $R^n$. G. Perelman in 1994 gave an astonishingly elegant/short proof of the Soul Conjecture: M is diffeomorphic to $R^n$ if it has positive curvature at only one point.
2. Gromov's Betti number theorem. There is a constant $C = C(n)$ such that if M is a compact connected n-dimensional Riemannian manifold with positive sectional curvature then the sum of its Betti numbers is at most C.
3. Grove–Petersen's finiteness theorem. Given constants C, D and V, there are only finitely many homotopy types of compact n-dimensional Riemannian manifolds with sectional curvature $K \geq C$, diameter $\leq D$ and volume $\geq V$.

Sectional curvature bounded above

1. The Cartan–Hadamard theorem states that a complete simply connected Riemannian manifold M with nonpositive sectional curvature is diffeomorphic to the Euclidean space $R^n$ with $n = \dim M$ via the exponential map at any point. It implies that any two points of a simply connected complete Riemannian manifold with nonpositive sectional curvature are joined by a unique geodesic.
2. The geodesic flow of any compact Riemannian manifold with negative sectional curvature is ergodic.
3. If M is a complete Riemannian manifold with sectional curvature bounded above by a strictly negative constant k then it is a CAT(k) space. Consequently, its fundamental group $\Gamma = \pi_1(M)$ is Gromov hyperbolic. This has many implications for the structure of the fundamental group:

- it is finitely presented;
- the word problem for Γ has a positive solution;
- the group Γ has finite virtual cohomological dimension;
- it contains only finitely many conjugacy classes of elements of finite order;
- the abelian subgroups of Γ are virtually cyclic, so that it does not contain a subgroup isomorphic to Z×Z.

Ricci curvature bounded below

1. Myers theorem. If a complete Riemannian manifold has positive Ricci curvature then its fundamental group is finite.
2. Bochner's formula. If a compact Riemannian n-manifold has non-negative Ricci curvature, then its first Betti number is at most n, with equality if and only if the Riemannian manifold is a flat torus.
3. Splitting theorem. If a complete n-dimensional Riemannian manifold has nonnegative Ricci curvature and a straight line (i.e. a geodesic that minimizes distance on each interval) then it is isometric to a direct product of the real line and a complete (n-1)-dimensional Riemannian manifold that has nonnegative Ricci curvature.
4. Bishop–Gromov inequality. The volume of a metric ball of radius r in a complete n-dimensional Riemannian manifold with positive Ricci curvature has volume at most that of the volume of a ball of the same radius r in Euclidean space.
5. Gromov's compactness theorem. The set of all Riemannian manifolds with positive Ricci curvature and diameter at most D is pre-compact in the Gromov-Hausdorff metric.[13,14,15]

Negative Ricci curvature

1. The isometry group of a compact Riemannian manifold with negative Ricci curvature is discrete.
2. Any smooth manifold of dimension $n \geq 3$ admits a Riemannian metric with negative Ricci curvature.[4] (This is not true for surfaces.)

Positive scalar curvature

1. The n-dimensional torus does not admit a metric with positive scalar curvature.
2. If the injectivity radius of a compact n-dimensional Riemannian manifold is $\geq \pi$ then the average scalar curvature is at most n(n-1).

## RESULTS

Geometric analysis on Riemannian manifolds combines methods from differential geometry and the theory of partial differential equations to investigate the interplay between curvature, topology and analytic properties of spaces. Central topics include the study of the Laplace–Beltrami operator and associated heat kernels, the analysis of curvature flows, index theory on spin and non-spin manifolds, and the resolution of variational problems such as the Yamabe and isoperimetric problems. Through curvature bounds and comparison theorems, one can derive global geometric consequences from local analytic estimates, for example relating Ricci curvature lower bounds to volume growth and eigenvalue estimates. Nonlinear equations such as

the prescribed scalar curvature equation and harmonic map equations yield deep insights into the moduli of metrics and the structure of manifolds. Techniques such as the calculus of variations, Sobolev inequalities and spectral theory provide a framework to address questions of existence, uniqueness and regularity of geometric structures. The field has broad applications ranging from general relativity and gauge theory to material science, where geometric constraints guide the behaviour of physical systems. Recent advances in synthetic curvature concepts and nonsmooth spaces have further expanded the scope of geometric analysis beyond the classical smooth category. [16,17,18]

## CONCLUSION

Riemannian geometry is characterized, and research is oriented towards and shaped by concepts (geodesics, connections, curvature, ... ) and objectives, in particular to understand certain classes of (compact) Riemannian manifolds defined by curvature conditions (constant or positive or negative curvature, ... ). By way of contrast, geometric analysis is a perhaps somewhat less system- atic collection of techniques, for solving extremal problems naturally arising in geometry and for investigating and characterizing their solutions. It turns out that the two fields complement each other very well; geometric analysis offers tools for solving difficult problems in geometry, and Riemannian geom- etry stimulates progress in geometric analysis by setting ambitious goals. It is the aim of this article to be a systematic and comprehensive intro- duction to Riemannian geometry and a representative introduction to the methods of geometric analysis. It attempts a synthesis of geometric and an- alytic methods in the study of Riemannian manifolds.[19,20]

## REFERENCES

1. Schoen, Richard; Yau, Shing Tung (2010). Lectures on Differential Geometry. International Press of Boston. ISBN 978-1-571-46198-8.
2. Andrews, Ben (2010). The Ricci Flow in Riemannian Geometry: A Complete Proof of the Differentiable 1/4-Pinching Sphere Theorem (1st ed.). Springer. ISBN 978-3-642-16285-5.
3. Jost, Jürgen (2005). Riemannian geometry and Geometric Analysis (4th ed.). Springer. ISBN 978-3-540-25907-7.
4. Lee, Jeffrey M. (2009). Manifolds and Differential Geometry. American Mathematical Society. ISBN 978-0-8218-4815-9.
5. Helgason, Sigurdur (2000). Groups and Geometric Analysis (Integral Geometry, Invariant Differential Operators and Spherical Functions) (2nd ed.). American Mathematical Society. ISBN 978-0-8218-2673-7.
6. Struik, D. J. "Outline of a History of Differential Geometry: I." Isis, vol. 19, no. 1, 1933, pp. 92–120. JSTOR, www.jstor.org/stable/225188.
7. Clairaut, A.C., 1731. Recherches sur les courbes à double courbure. Nyon.
8. O'Connor, John J.; Robertson, Edmund F., "Leonhard Euler", MacTutor History of Mathematics Archive, University of St Andrews
9. Spivak, M., 1975. A comprehensive introduction to differential geometry (Vol. 2). Publish or Perish, Incorporated.
10. Gauss, C.F., 1828. Disquisitiones generales circa superficies curvas (Vol. 1). Typis Dieterichianis.

11. Struik, D.J. "Outline of a History of Differential Geometry (II)." Isis, vol. 20, no. 1, 1933, pp. 161–191. JSTOR, www.jstor.org/stable/224886
12. O'Connor, John J.; Robertson, Edmund F., "Non-Euclidean Geometry", MacTutor History of Mathematics Archive, University of St Andrews
13. Milnor, John W., (1982) Hyperbolic geometry: The first 150 years, Bull. Amer. Math. Soc. (N.S.) Volume 6, Number 1, pp. 9–24.
14. 1868 On the hypotheses which lie at the foundation of geometry, translated by W.K.Clifford, Nature 8 1873 183 – reprinted in Clifford's Collected Mathematical Papers, London 1882 (MacMillan); New York 1968 (Chelsea) http://www.emis.de/classics/Riemann/. Also in Ewald, William B., ed., 1996 "From Kant to Hilbert: A Source Book in the Foundations of Mathematics", 2 vols. Oxford Uni. Press: 652–61.
15. Christoffel, E.B. (1869). "Ueber die Transformation der homogenen Differentialausdrücke zweiten Grades". Journal für die Reine und Angewandte Mathematik. 70.
16. Ricci, Gregorio; Levi-Civita, Tullio (March 1900). "Méthodes de calcul différentiel absolu et leurs applications" [Methods of the absolute differential calculus and their applications]. Mathematische Annalen (in French). 54 (1–2). Springer: 125–201. doi:10.1007/BF01454201. S2CID 120009332.
17. Dieudonné, J., 2009. A history of algebraic and differential topology, 1900-1960. Springer Science & Business Media.
18. Fré, P.G., 2018. A Conceptual History of Space and Symmetry. Springer, Cham.
19. The area preserving condition (or the twisting condition) cannot be removed. If one tries to extend such a theorem to higher dimensions, one would probably guess that a volume preserving map of a certain type must have fixed points. This is false in dimensions greater than 3.